ÉRETTSÉGI VIZSGA ● 2006. május 9.

MATEMATIKA ANGOL NYELVEN MATHEMATICS

EMELT SZINTŰ ÍRÁSBELI ÉRETTSÉGI VIZSGA ADVANCED LEVEL WRITTEN EXAM

JAVÍTÁSI-ÉRTÉKELÉSI ÚTMUTATÓ KEY AND GUIDE FOR EVALUATION

OKTATÁSI MINISZTÉRIUM
MINISTRY OF EDUCATION

Important Information

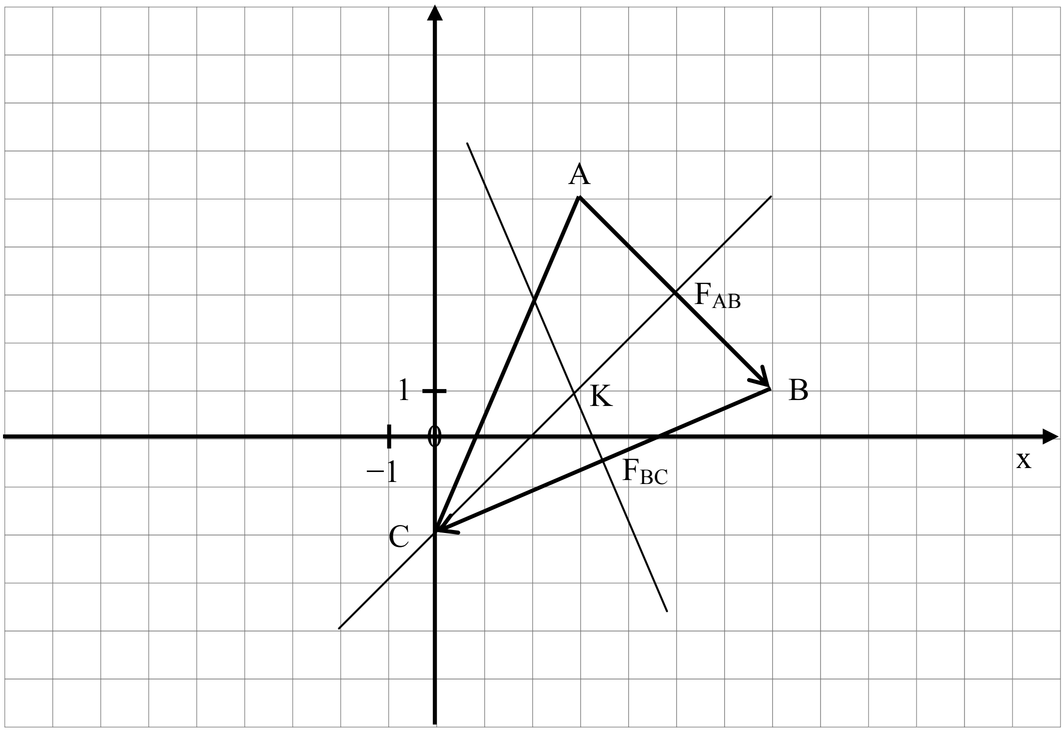
Formal requirements:

- The papers must be assessed in **pen and of different colour** than the one used by the candidates. Errors and flaws should be indicated according to ordinary teaching practice.
- The first one among the shaded rectangles next to each question contains the maximal score for that question. The **score** given by the examiner should be entered into the other **rectangle**.
- **In case of correct solutions**, it is enough to enter the maximal score into the corresponding rectangle.
- In case of faulty or incomplete solutions, please indicate the corresponding partial scores within the body of the paper.

Substantial requirements:

- In case of some problems there are more than one solutions outlined with the corresponding marking schemes. However, if you happen to come across with some **solution different** from those in the assessment bulletin, please identify the parts equivalent to those in the solution provided here and do your marking accordingly.
- The scores in this evaluation guide can be split further. Keep in mind, however, that the number of points awarded for any item can be an integer number only.
- In case of a correct answer and a valid argument the maximal score can be awarded even if the actual solution is **less detailed** than that in this guide.
- If there is a **calculation error** or any other flaw in the solution, then the score should be deducted for the actual item where the error has occured. If the candidate is going on working with the faulty intermediate result and the problem has not been changed essentially due to the error, then the subsequent partial scores should be given out.
- If there is a **fundamental error** within an item (these are separated by double lines in this bulletin), even formally correct steps should not be awarded by any points, whatsoever. However, if the candidate is using the wrong result obtained by the invalid argument throughout the subsequent steps correctly, they should be given the maximal score for the remaing parts if the problem has not been changed essentially due to the error.
- If a **measuring unit** occurs in braces in this bulletin, the solution is complete even if the candidate does not indicate this unit.
- If there are more than one attempts to solve a problem, there is **just one** of them (with the highest score) that can be considered in the final assesment.
- You should **not give out any bonus points** (points beyond the maximal score for a solution or for some part of the solution).
- You should not reduce the score for erroneous calculations or steps unless its results are actually used by the candidate in the course of the solution.
- **There are only 4 questions to be marked out of the 5 in part II of this exam paper.** Hopefully, the candidate has entered the number of the question not to be marked in the square provided for this purpose. Accordingly, this question should not be assessed even if there is some kind of solution written down in the paper. Should there be any ambiguity about the student's request with respect to the question not to be checked, it is the last one in this problem set, by default, that should not be marked.

I.

1. a)		
		
Vertex C of the triangle is intercepted from the y –axis by the perpendicular bisector of the segment AB . The midpoint of AB is $F_{AB}(5,3)$.	1 point	
A normal vector of the perpendicual bisector of AB is $\overrightarrow{AB}(4,-4)$.	1 point	
The equation of the perpendicual bisector of AB is: $x-y=2$.	1 point	
The vertex opposite to the base AB is $C(0,-2)$.	1 point	*If the candidate makes a correct guess based on a neat diagram about the coordinates of the vertex C without any further calculations then at most 2 of the above 4 points can be given.*
Total:	**4 points**	
b)		
The circumcentre is the common point of the perpendicular bisector of the base AB and that of one of the sides.	1 point	*This point may be given if the argument is clear from the calculation.*
The midpoint of the side BC is $F_{BC}\left(\frac{7}{2},-\frac{1}{2}\right)$.	1 point	
A normal vector of the perpendicular bisector of BC	1 point	

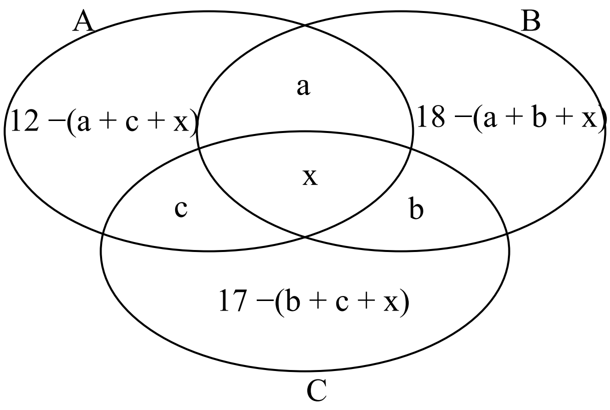
is $\overrightarrow{CB}(7,3)$.		
The equation of the perpendicular bisector of BC is $7x+3y=23$.	1 point	
Solving the simultaneous system formed by the equations of the perpendicular bisectors of AB and that of BC, respectively, yields $x=2.9$; $y=0.9$, and thus the circumcentre is $K(2.9,\ 0.9)$.	2 points	
The square of the circumradius is $r^2=KC^2=2\cdot 2.9^2=16.82$.	1 point	
The equation of the circumcircle is $(x-2.9)^2+(y-0.9)^2=16.82$.	1 point	
Total:	**8 points**	

2.		
Denote the length of the edges of the red and the blue cube by a and b, respectively. The surface area of the red cube is $6a^2$ and that of the blue one is $6b^2$.	2 points	
By the condition we have $6a^2=\frac{3}{4}\cdot 6b^2$.	3 points	
Therefore, using the fact that $a>0$ and $b>0$ one gets $a=\frac{\sqrt{3}}{2}b$.	2 points	
Expressing the volume of the red cube in terms of that of the blue one yields $a^3=\frac{3\sqrt{3}}{8}b^3$.	3 points	
Since $\frac{3\sqrt{3}}{8}\approx 0.65$, the volume of the red cube is $\approx 65\%$ of that of the blue cube.	1 point	
Therefore, the volume of the red cube is about 35% less than that of the blue cube.	1 point	
Total:	**12 points**	

3. a)		
If the roots of the equation $x^2-x+p=0$ are x_1, x_2, then x_1+1, x_2+1 are those of the equation $x^2+px-1=0$.	2 points	*These points may also be given if the candidate writes down the roots in parametric form with the help of the quadratic formula*
By the Viéte-formula for the sum of the roots in the equations $x_1+x_2=1$ and $(x_1+1)+(x_2+1)=-p$.	3 points	*These 3 points may also be given if the roots obtained by the quadratic formula are matched correctly.*

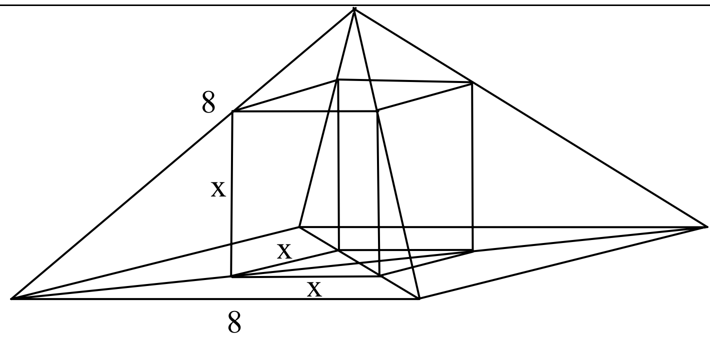
Therefore, the only possible value of p is -3.	3 points	
If $p=-3$ then both equations have real roots.	1 point	*This 1 point may also be given if the candidate is checking the discriminant.*
Total:	**9 points**	
b)		
The discriminant of the equation $x^2-x+5=0$ is negative and thus it has no real roots.	2 points	
The roots of the equation $x^2+5x-1=0$ are $x_1=\frac{-5+\sqrt{29}}{2}(\approx 0.19)$; $x_2=\frac{-5-\sqrt{29}}{2}(\approx -5.19)$.	2 points	
Total:	**4 points**	

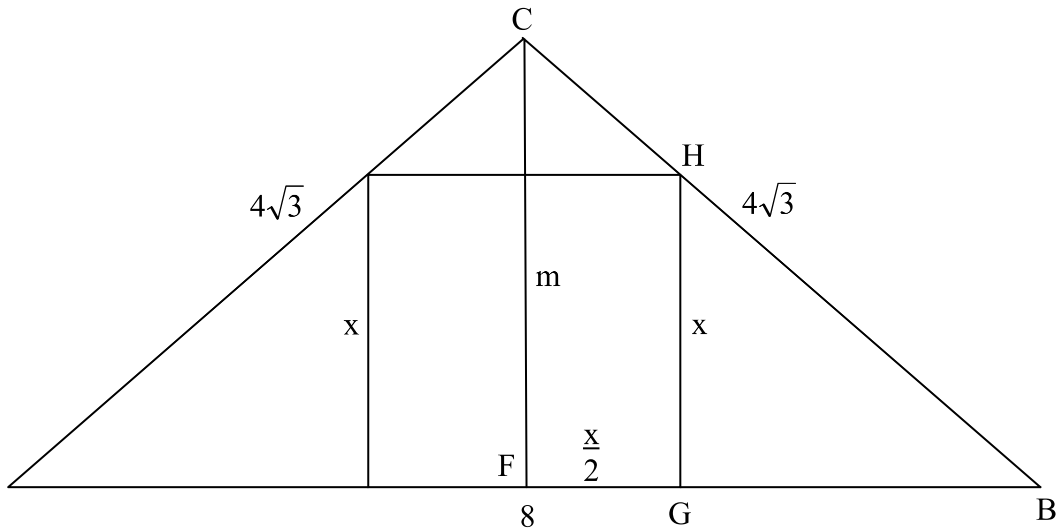
4. a) (1st. solution)		
Denote the set of scientists publishing in the field of research, education and communication by *A*, *B* and *C*, respectively. The conditions of the question hence can be written as $\lvert A\rvert=12,\quad \lvert B\rvert=18,\quad \lvert C\rvert=17,\quad \lvert A\cup B\cup C\rvert=30.$	1 point	
$\lvert A\cap B\rvert+\lvert B\cap C\rvert+\lvert C\cap A\rvert-3\cdot\lvert A\cap B\cap C\rvert=7.$	2 points	
By virtue of the sieve-formula $30=\lvert A\cup B\cup C\rvert=$ $=\lvert A\rvert+\lvert B\rvert+\lvert C\rvert-\lvert A\cap B\rvert-\lvert B\cap C\rvert-\lvert C\cap A\rvert+\lvert A\cap B\cap C\rvert=$ $=12+18+17-7-2\cdot\lvert A\cap B\cap C\rvert.$	3 points	
Therefore, $\lvert A\cap B\cap C\rvert=5$.	2 points	
The probability in question is hence $P=\frac{5}{30}=\frac{1}{6}$.	2 points	
Total:	**10 points**	

a) (2nd solution)		
	3 points	
By the conditions (1) $a+b+c=7$.	1 point	
(2) $x+a+b+c+12-(a+c+x)+18-(a+b+x)+$ $+17-(b+c+x)=30$	2 points	
Collecting the terms on the l.h.s. of (2) $47-2x-(a+b+c)=30$.	1 point	
Substituting (1) yields $x=5$.	1 point	
The probability in question is hence $P=\frac{5}{30}=\frac{1}{6}$.	2 points	
Total:	**10 points**	
b)		
There are 5 scientists publishing in each of the three topics, 7 scientists publising in exactly two of the topics and thus there are 12 scientists altogether, who have been publishing in at least two of the given topics.	2 points	
The number of specialists is hence $30-12=18$.	2 points	
Total:	**4 points**	

II.

5. a)



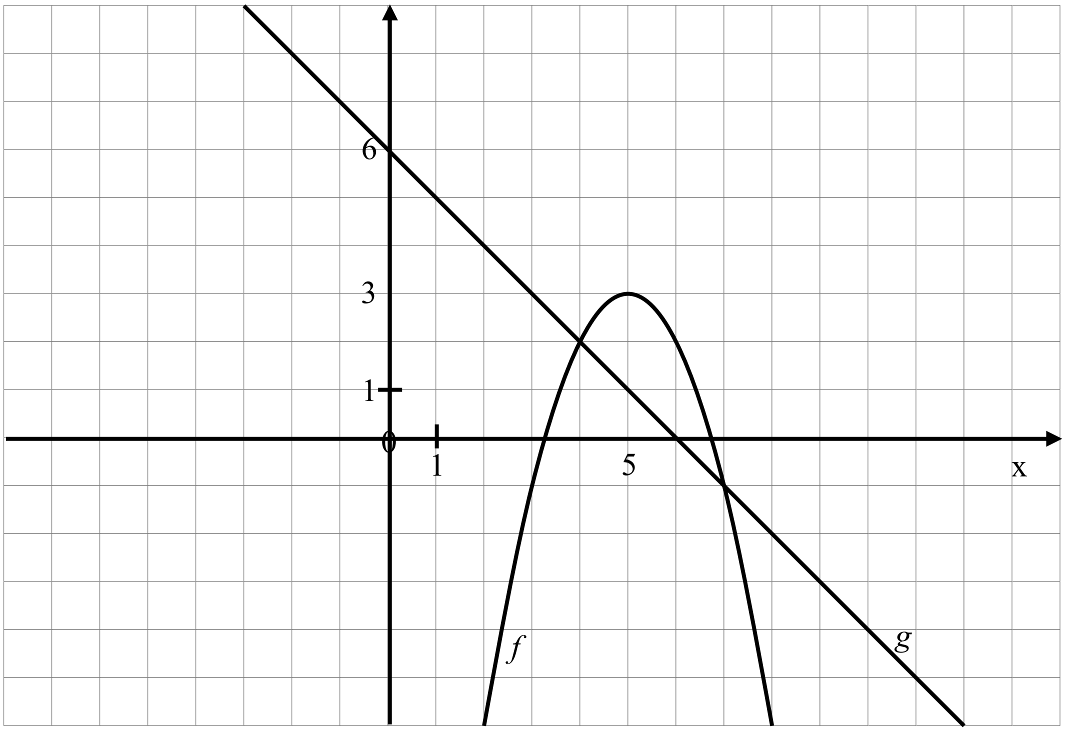


Consider, for the calculation of the edge of the room in the roof, the planar section of the pyramid through its vertex and the midpoints of two opposite edges.	2 points	*These 2 points may also be given if the diagram shows a clear visual perception of the spatial objects of the question.*
This planar section is an isosceles triangle of base 8 meters and edges $4\sqrt{3}$ meters long, respectively.	1 point	
By Pithagoras' theorem the altitude dropped onto the base of this triangle is $m = 4\sqrt{2}$ (m).	1 point	
Using the labellings of the diagram, the right triangles *CFB* and *HGB* are similar,	1 point	
because their common acute angle is *FBC*.	1 point	
If x denotes the length of the edge of the cube (it is the length of the side of the square in the planar section) then, by similarity, $\frac{x}{4-\frac{x}{2}} = \frac{4\sqrt{2}}{4} = \sqrt{2}$,	1 point	

and hence $x=\frac{8\sqrt{2}}{2+\sqrt{2}}\approx 3.3\,(\text{m})$.	1 point	
The area of the base of the room is $T=x^2=\frac{64}{3+2\sqrt{2}}\approx 11\,\text{m}^2$.	1 point	
Total:	**9 points**	
b)		
From the previous results the height of the pyramid is $m=4\sqrt{2}$.	1 point	
The volume of the roof (in fact, it is the pyramid) is hence $V_r=\frac{8^2\cdot 4\sqrt{2}}{3}\,\text{m}^3=\frac{256\sqrt{2}}{3}\,\text{m}^3\approx 120.68\,\text{m}^3$.	2 points	*The corresponding points are due even if the approxiamate values are not written down.*
The volume of the cube is $V_c=\left(\frac{8\sqrt{2}}{2+\sqrt{2}}\right)^3\text{m}^3=\frac{1024\sqrt{2}}{\left(2+\sqrt{2}\right)^3}\,\text{m}^3\approx 36.38\,\text{m}^3$.	2 points	
The ratio of the two volumes is hence $\frac{V_c}{V_r}=\frac{12}{\left(2+\sqrt{2}\right)^3}\approx 0.3015$.	1 point	
Thus the room fills approximately 30% of the airspace.	1 point	
Total:	**7 points**	

6. a)		
The equation to be solved is $-x^2+10x-22=-x+6$. Collecting the terms: $x^2-11x+28=0$.	1 point	
The solutions are $x_1=4, x_2=7$.	2 points	
Total:	**3 points**	
b)		
The slopes of the tangents at the points of intersection are $m_1=f'(x_1)$ and $m_2=f'(x_2)$, respectively. $f'(x)=-2x+10$.	1 point	
Hence $m_1=f'(4)=2$ and $m_2=f'(7)=-4$.	2 points	
The two graphs intersect at $M_1(4,2)$ and $M_2(7,-1)$.	2 points	
The equations of the corresponding tangents are $e_1:\ y-2=2(x-4)$ that is $y=2x-6$,	1 point	*The corresponding points may be given for any correct form of the equations of the tangents.*
$e_2:\ y+1=-4(x-7)$, that is $y=-4x+27$.	1 point	
Total:	**7 points**	

c)



Sketching the graphs of *f* and *g*.	1 point	*The candidate may get at most 4 points if there are errors in the graphs or there are wrong numbers used (the limits of integration are not correct, for example). The corresponding marks should also be given if the candidate finds the result as the difference of the two separately calculated integrals or instead of integrating the function g the area of the corresponding isosceles right triangle is subtracted from the integral of f.*
The area of the given region is $T=\int_4^6 f(x)dx-\int_4^6 g(x)dx=\int_4^6 (f(x)-g(x))dx$.	1 point	
Since $f(x)-g(x)=-x^2+11x-28$ the area is equal to $T=\int_4^6 (-x^2+11x-28)dx=\left[-\frac{x^3}{3}+11\frac{x^2}{2}-28x\right]_4^6=$	2 points	
$=\left(-\frac{6^3}{3}+11\cdot\frac{6^2}{2}-28\cdot 6\right)-\left(-\frac{4^3}{3}+11\cdot\frac{4^2}{2}-28\cdot 4\right)=\frac{10}{3}$.	2 points	
Total:	**6 points**	

7. a)		
Let the lengths of the distance in kms between Szeged - Cegléd and Cegléd - Budapest be s_1 and s_2, respectively and denote the original average velocity in km/h of the train by *v*. The travelling time in hours of the train on Monday is hence $\frac{s_1}{v}+\frac{3s_2}{v}$.	2 points	*A brief argument without formulae similar to the one below can also be accepted as a correct solution. The 30 minutes difference relative to the weekend trip is due to the twice bigger weekday velocity along the 19 kilometers line. Hence the velocity of the train is the double of 19/0.5 that is 76 km/h.*
The weekend travelling time also in hours is $\frac{s_1+19}{v}+\frac{3(s_2-19)}{v}$.	3 points	
According to the condition about the difference of the two travelling times one can write $\left(\frac{s_1}{v}+\frac{3s_2}{v}\right)-\left(\frac{s_1+19}{v}+\frac{3(s_2-19)}{v}\right)=\frac{1}{2}$.	3 points	
Solving the equation one gets the average velocity of the train: it is $v=76\,\text{km/h}$.	2 points	
Total:	**10 points**	

b)

Type of the ticket	Full price	20% discount	33% discount	50% discount	67,5% discount	75% discount	90% discount	95% discount	free
No. of passangers	84	18	44	110	11	35	31	29	38
Actual Price (Ft)	**2000**	**1600**	**1340**	**1000**	**650**	**500**	**200**	**100**	**0**

Filling the table correctly	2 points	*If there are faulty results among the actual prices but no more than four of them, then there can be at most 1 point given. If there are more than four faulty results then no point can be given.*

The average ticket price in forints is

$$\frac{84\cdot 2000+18\cdot 1600+44\cdot 1340+110\cdot 1000+11\cdot 650+35\cdot 500+31\cdot 200+29\cdot 100+38\cdot 0}{400}=$$

$=\frac{399510}{400}=998.775(\approx 999\text{ Ft or }1000\text{ Ft})$.	2 points	*The 2 points are due even if there are some wrong ones among the actual ticket prices but the method of*

		calculating their average is correct.
This is approxiamtely 50% of the full price and thus the discount on the average ticket price would be 50%.	2 points	*The 2 points are due even if the candidate has performed correct calculations based on faulty data or if its different result is due to various rounding effects.*
Total:	**6 points**	
8. a)		
$\overline{a}$, $\overline{ab}$, $\overline{bba}$ are the consecutive terms of an arithmetic progression if and only if $\overline{bba}-\overline{ab}=\overline{ab}-\overline{a}$.	1 point	
Switching to decimal notation one gets $(110b+a)-(10a+b)=(10a+b)-a$,	1 point	
Simplifying we obtain $a=6b$.	1 point	
Since *a* and *b* are decimal digits, $a=6, b=1$.	2 points	
The three numbers are hence 6; 61; 116, and the common difference is 55.	1 point	
The sum of the first one hundred terms is $S_{100}=\frac{100}{2}(2\cdot 6+99\cdot 55)=272850$.	1 point	
Total:	**7 points**	
b)		
The first term of the geometric progression is *a* and its common ratio is *q*. If $q=1$ then the progression is constant and thus the corresponding sums are all equal, the three identical numbers are the consecutive terms of a geometric progression..	1 point	
If $q\neq 1$, then the sum of the first *n* terms is $S_n^{(1)}=a\cdot\frac{q^n-1}{q-1}$.	1 pont	
The sum of the second *n* terms is $S_n^{(2)}=aq^n\cdot\frac{q^n-1}{q-1}$.	2 points	
The sum of the third *n* terms is $S_n^{(3)}=aq^{2n}\cdot\frac{q^n-1}{q-1}$.	2 points	
It is necessary and sufficient for these sums to form a geometric progression in this order if $\left(S_n^{(2)}\right)^2=S_n^{(1)}\cdot S_n^{(3)}$ holds.	1 point	
In fact, this is the case, since	2 points	

$S_n^{(1)} \cdot S_n^{(3)} = a^2 q^{2n} \cdot \left(\frac{q^n - 1}{q - 1}\right)^2 = \left(aq^n \cdot \frac{q^n - 1}{q - 1}\right)^2 = \left(S_n^{(2)}\right)^2.$		
Total:	**9 points**	*The last 3 point may be given for any correct argument.*

9. a)		
If the first two numbers are a and b ($a < b$), then the third number is $a + b$ and the fourth one is $2(a + b)$.	1 point	
By condition, we have $2(a + b) \leq 40$ that is $a + b \leq 20$.	1 point	
Here $a < b$ implies $a \leq 9$, that is the smallest number can be at most 9.	2 points	
Total:	**4 points**	
b)		
There are two possible such quadruples.	2 points	
9, 10, 19, 38;	1 points	
9, 11, 20, 40.	1 points	
Total:	**4 points**	
c)		
The set of tickets filled by Andrew's rule can be grouped according to the choice of the first number. The first number is 1 2 3 4 5 6 7 8 9.	1 point	
The number of tickets are respectively: 18 16 14 12 10 8 6 4 2.	2 points	
The number of different tickets are hence $2 + 4 + ... + 18 = 90$.	1 point	
The number of quadruples that can be selected from the first 40 positive integers is $\binom{40}{4} = 91390$.	2 points	
The probability of a bingo is hence $P = \frac{90}{91390} \approx 9{,}85 \cdot 10^{-4}$.	2 points	
Total:	**8 points**	